Generalized Maximum Specific Range Performance

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Generalized criteria for optimum cruise (point) performance in quasisteady flight are derived for jet and propeller aircraft, using logarithmic differentiation. The derivation is valid not only for low speeds, but also for Mach numbers where compressibility effects are present; and it is not based on the usual assumption of constant TSFC or constant overall powerplant efficiency. Optimum conditions are obtained for the unconstrained variation of altitude and Mach number and for the case where constraints are imposed on W/δ or T/δ. The resulting criteria define the relationships between the logarithmic partial derivatives of C_D with respect to C_L and M and of the overall powerplant efficiency with respect to M and the engine rating (thrust). Special conditions are considered, indicating that simple solutions can be given, e.g., for the optimum drag rise dC_D/dM and for aircraft with parabolic drag polars. The relationship with classical criteria for optimum long-range flight are also shown, and a numerical example for a high-subsonic jet transport illustrates a practical computational procedure. The present method is useful in the preliminary design phase, but may be used for on-board calculation of optimum flight conditions as well.

Nomenclature

C_D	= drag coefficient
C_{D_L}	= logarithmic partial derivative of C_D with respect to (wrt) C_L
C_{D_M}	= logarithmic partial derivative of C_D wrt M
$C_{D_{MD}}$	= C_D for minimum drag condition
C_L	= lift coefficient
$C_{L_{MD}}$	= C_L for minimum drag condition
C_T	= specific fuel consumption (TSFC)
C_1, C_2, C_3	= constants in Eqs. (28) and (29) for C_T
D	= drag (no index: total airplane drag)
D_{ind}	= induced drag
F	= fuel mass flow
H	= heating value of fuel
L	= lift
M	= flight Mach number
N	= engine speed, rpm
n	= exponent in Eq. (29) for C_T
P	= range parameter
p	= atmospheric pressure
p_0	= p at sea level ISA
S	= wing area
T	= net propulsive thrust
T_{to}	= T at sea level, static, takeoff
V	= true airspeed
W	= aircraft all-up-weight (AUW)
γ	= ratio of specific heats for air ($\gamma = 1.4$)
δ	= relative atmospheric pressure (p/p_0)
η	= overall powerplant efficiency
η_M	= logarithmic partial derivative of η wrt M
η_T	= logarithmic partial derivative of η wrt T/δ
θ	= relative atmospheric temperature

Introduction

In classical flight mechanics the optimum cruise performance of jet aircraft is usually considered on the basis of simplified assumptions of airplane and engine characteristics, e.g., no compressibility effects on the drag polar and constant specific fuel consumption. Because of such simplifications an unconstrained optimum is not obtained for the specific range. However, constrained optima for a given altitude or a given engine speed can be derived, resulting in the well-known optimum speed ratios of $\sqrt[4]{3}$ and $\sqrt[4]{2}$ with respect to the minimum drag speeds (cf., Refs. 1-3).

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In the present study a generalized approach is given to the problem of the maximum specific range of a jet aircraft. The airplane drag and engine performance data are specified in new logarithmic parameters, which enable the derivation of simple generalized conditions for optimum cruise with and without constraints. A numerical example is given in the Appendix for a high-subsonic jet aircraft; however, the method is also applicable to propeller and supersonic aircraft.

Basic Equations

The specific range V/F can be expressed in terms of the airplane and engine characteristics by the introduction of the overall powerplant efficiency,†

$$\eta = TV/HF \tag{1}$$

Since the specific range is considered for a given airplane all-up weight (AUW) and a given type of fuel, the range parameter P to be optimized for maximum specific range can be written as

$$P = \frac{W}{H}\frac{V}{F} = \eta\frac{W/\delta}{T/\delta} \tag{2}$$

This expression is to be maximized for quasisteady and quasihorizontal flight with the following equilibrium conditions, written in the usual way,

$$W/\delta = L/\delta = \tfrac{1}{2}\gamma p_0 M^2 S C_L \tag{3}$$

$$T/\delta = D/\delta = \tfrac{1}{2}\gamma p_0 M^2 S C_D \tag{4}$$

Under these conditions the range parameter according to Eq. (2) is identical to

$$P = \eta(L/D) \tag{5}$$

and represents the product of the overall powerplant efficiency and the aircraft aerodynamic efficiency. The object of the cruise point optimization is therefore to find a maximum of these combined efficiencies. Since both are mutually unrelated functions of cruise speed and altitude, the combined optimum does generally not coincide with any of the maxima for η and L/D.

†The term propulsive efficiency is used frequently in the literature, but may lead to confusion with propulsive jet efficiency, which is used in conjunction with thermal efficiency.

Neglecting the effect of Reynolds number variation, the aircraft drag polars are given by

$$C_D = f(C_L, M) \tag{6}$$

The overall powerplant efficiency can be derived from generalized engine performance data (turbojet and turbofan engines)

$$T/\delta = f[M, (N/\sqrt{\theta})] \tag{7a}$$

$$F/\delta\sqrt{\theta} = f[M, (N/\sqrt{\theta})] \tag{7b}$$

Elimination of the corrected engine speed $N/\sqrt{\theta}$ yields

$$F/\delta\sqrt{\theta} = f[M, (T/\delta)] \tag{8}$$

Hence, according to Eq. (1),

$$\eta = \frac{(T/\delta)(V/\sqrt{\theta})}{H(F/\delta\sqrt{\theta})} = f\left(M, \frac{T}{\delta}\right) \tag{9}$$

It can be shown[5] that this result holds for propeller aircraft as well, provided the overall powerplant efficiency is defined as the product of engine efficiency and propeller efficiency. From Eqs. (3), (4), (6), and (9) it is obvious that the range parameter P in Eq. (2) depends on two independent variables, e.g., C_L and M. For an unconstrained maximum of P the partial derivatives of this parameter with respect to both independent variables must be equal to zero.

Analysis of Derivatives

Using logarithmic differentiation of Eq. (2), it is found that the optimum range parameter has to satisfy the condition,

$$\mathrm{dlog}P = \mathrm{dlog}\eta + \mathrm{dlog}W/\delta - \mathrm{dlog}T/\delta = 0 \tag{10}$$

The terms of this equation can be derived from the basic equations given above. Using Eq. (9) one can write

$$\mathrm{d}\eta = \frac{\partial\eta}{\partial M}\mathrm{d}M + \frac{\partial\eta}{\partial T/\delta}\mathrm{d}\frac{T}{\delta}$$

or

$$\frac{\mathrm{d}\eta}{\eta} = \frac{\partial\eta}{\eta}\frac{M}{\partial M}\frac{\mathrm{d}M}{M} + \frac{\partial\eta}{\eta}\frac{T/\delta}{\partial T/\delta}\frac{\mathrm{d}T/\delta}{T/\delta}$$

Hence:

$$\mathrm{dlog}\eta = \frac{\partial\log\eta}{\partial\log M}\mathrm{dlog}M + \frac{\partial\log\eta}{\partial\log T/\delta}\mathrm{dlog}\frac{T}{\delta}$$

$$= \eta_M \mathrm{dlog}M + \eta_T \mathrm{dlog}\frac{T}{\delta} \tag{11}$$

In this expression two engine parameters are introduced which characterize the variation of the total powerplant efficiency with M and T/δ, respectively,

$$\eta_M = \left(\frac{\partial\log\eta}{\partial\log M}\right)_{T/\delta} \qquad \eta_T = \left(\frac{\partial\log\eta}{\partial\log T/\delta}\right)_M \tag{12}$$

Figure 1 shows the characteristics of a typical high-bypass turbofan engine; the parameters η_M and η_T can be determined from the slopes of the curves. (In all numerical examples the logarithm with basenumber 10 is used.) From the equilibrium conditions [Eqs. (3) and (4)] one finds in the same way,

$$\mathrm{dlog}W/\delta = \mathrm{dlog}C_L + 2\mathrm{dlog}M \tag{13}$$

and

$$\mathrm{dlog}T/\delta = \mathrm{dlog}C_D + 2\mathrm{dlog}M \tag{14}$$

Moreover, from the aircraft drag polar

$$\mathrm{dlog}C_D = \frac{\partial\log C_D}{\partial\log C_L}\mathrm{dlog}C_L + \frac{\partial\log C_D}{\partial\log M}\mathrm{dlog}M$$

$$= C_{D_L}\mathrm{dlog}C_L + C_{D_M}\mathrm{dlog}M \tag{15}$$

This equation introduces two proposed drag parameters of the aircraft, defined as

$$C_{D_L} = \left(\frac{\partial\log C_D}{\partial\log C_L}\right)_M \qquad C_{D_M} = \left(\frac{\partial\log C_D}{\partial\log M}\right)_{C_L} \tag{16}$$

An example of aerodynamic data on a log base is given in Fig. 2. The slopes of these curves define the value of C_{D_L} and C_{D_M}; as an example, $C_{D_L}=1$ and $C_{D_M}=1$ have been identified.

After substitution of Eqs. (11) and (13-15) into Eq. (10) the condition for the maximum range parameter is obtained,

$$\mathrm{dlog}P = (\eta_T C_{D_L} - C_{D_L} + 1)\mathrm{dlog}C_L$$

$$+ (\eta_T C_{D_M} + 2\eta_T + \eta_M - C_{D_M})\mathrm{dlog}M = 0 \tag{17}$$

This equation will be evolved further for the case where an unconstrained variation of C_L and M occurs and for other cases where constraints are imposed on combinations of these variables.

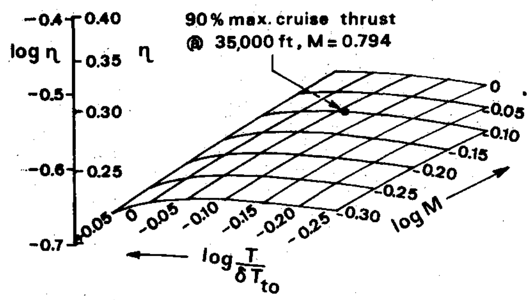

Fig. 1 Logarithmic presentation of engine total efficiency for a high-bypass engine.

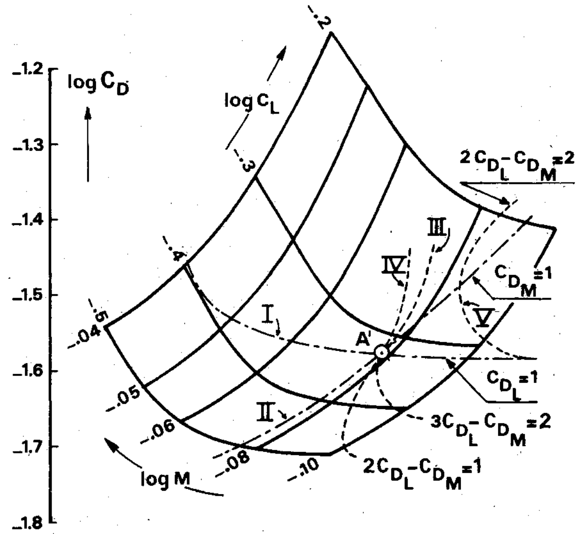

Fig. 2 Logarithmic presentation of drag polars and derivatives C_{D_L} and C_{D_M}. Note: Point A′ and lines I-IV define optimum conditions for $\eta_M=1$, $\eta_T=0$.

Conditions for Maximum Specific Range

Unconstrained or Absolute Maximum

A partial optimum for the range parameter is obtained for given Mach number (dlog $M=0$) from the first term between the brackets in Eq. (17), defining the optimum C_L in terms of the logarithmic derivative C_{D_L},

$$C_{D_L} = 1/(1-\eta_T) \tag{18}$$

Alternatively, a partial maximum for a given C_L follows from the second bracketed term in Eq. (17),

$$C_{D_M} = (2\eta_T + \eta_M)/(1-\eta_T) \tag{19}$$

Equations (18) and (19) define the partial optima in terms of the slopes of logC_D vs logC_L and log$M$, respectively (cf. Fig. 2). Therefore, for given engine characteristics η_M and η_T, these optima may be allocated in the drag polars in the form of two curves identifying the slopes determined by Eqs. (18) and (19).

Figure 3, based on Figs. 1 and 2, shows an example of the range parameter P, plotted vs M and C_L in the form of isoparameter curves. Although for the present analysis it is not necessary to calculate these curves for P, they are shown here for illustrative purposes.

In Fig. 3 Eq. (18) is represented by line I interconnecting the points where the isoparameter curves for P are vertical. Similarly, line II represents Eq. (19) and connects the points where these curves are horizontal.

The *unconstrained* or *absolute* maximum for the specific range is given by the intersection of the two curves I and II (point A) for which Eqs. (18) and (19) are satisfied simultaneously. Accordingly, altitude and engine rating at a given aircraft weight, corresponding to point A, are fixed according to Eqs. (3) and (4).

Constrained Maxima

In operational practice it is not always feasible nor desirable to fly at the maximum value of P, indicated by point A in Fig. 3. Practical considerations may entail preselection of the altitude or flight with a constant engine rating (e.g., maximum cruise). Also, buffet-free flight in gusty conditions may force the pilot to reduce the cruise lift coefficient relative to the optimum by flying below the optimum altitude. These two constraints deserve special attention and will be considered below.

Constraint on W/δ (or $C_L M^2$)

This case defines cruising at given altitude in the case of a specified aircraft weight. Applying this constraint to Eq. (13) we find

$$\text{dlog}C_L = -2\text{dlog}M \tag{20}$$

Substitution of Eq. (20) into Eq. (17) yields the condition for maximum specific range after some algebraic rearrangements,

$$2C_{D_L} - C_{D_M} = 2 - \eta_M/(1-\eta_T) \tag{21}$$

This result is indicated in Fig. 3 by line III. The intersection with the specified constraint $C_L M^2 = $ const in this graph represents the optimum flight condition, e.g., point B.

Constraint on T/δ (or $C_D M^2$)

The significance of the constraint $T/\delta = $ const appears from the observation that for low- and medium-bypass-ratio jet engines at high altitudes and high-subsonic speeds T/δ is approximately invariable with speed and altitude for constant engine speed (rating). Therefore, this constrained optimum yields the best cruise condition *for given engine rating*.

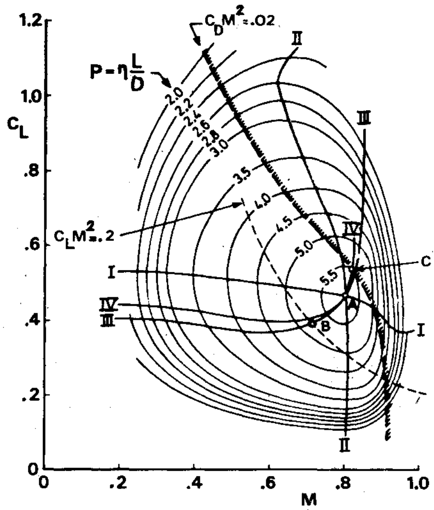

Fig. 3 Isoparameter plot for a high-subsonic transport and optimum cruise conditions.

Combination of Eqs. (14) and (15) leads to the relationship,

$$\text{dlog}C_L = -(1/C_{D_L})(C_{D_M}+2)\text{dlog}M \tag{22}$$

From Eq. (17) the condition for maximum range parameter is obtained after some algebraic manipulations

$$C_{D_L}(2+\eta_M) - C_{D_M} = 2 \tag{23}$$

This result is given in Fig. 3 by line IV, determining the optimum by intersection with the curve $C_D M^2 = $ const (e.g., point C).

Since the specific range depends on two independent variables only, all lines I-IV in Fig. 3 intersect at the same point A. This can easily be verified by substitution of Eqs. (18) and (19) in either Eq. (21) or (23), which leads to an identity.

The criteria for maximum specific range derived in this section are related to point performance. However, considering cruise techniques where variations in aircraft weight are involved, the present results may be interpreted to define overall optimum cruise techniques, provided the values of C_L and M (and hence W/δ and T/δ) are kept constant during the flight. This results in a quasisteady cruise/climb technique (with constant engine speed in the stratosphere).

Comments on the Logarithmic Derivatives

The physical significance of logarithmic partial derivatives becomes clear when it is noted that, for example,

$$C_{D_M} = \frac{\partial \log C_D}{\partial \log M}\bigg|_{C_L} = \frac{M}{C_D}\frac{\partial C_D}{\partial M}\bigg|_{C_L} \approx \frac{\Delta C_D/C_D}{\Delta M/M}\bigg|_{C_L}$$

Therefore C_{D_M} represents the percentage change in C_D for a given percentage change in Mach number at constant C_L. The use of these logarithmic derivatives leads to a considerable simplification of the equations. Another advantage is that their numerical values are convenient to use, as appears from the following discussion.

Aerodynamic Derivatives

The numerical values of the aerodynamic derivatives C_{D_L} and C_{D_M} can be obtained from the drag polars, defined by polynomials or in graphic form, preferably on a log base (Fig. 2). The derivative C_{D_M} is a direct measure of the drag rise and is approximately equal to zero at subcritical speeds.

For parabolic drag polars C_{D_L} is directly related to the ratio of induced to total airplane drag,

$$C_{D_L} = \frac{C_L}{C_D}\frac{dC_D}{dC_L} = \frac{2C_L^2}{C_D}\frac{dC_D}{dC_L^2} = 2\frac{D_{ind}}{D} \quad (24)$$

Alternatively, the lift coefficient C_L for a given C_{D_L} is related to the lift coefficient for minimum drag $C_{L_{MD}}$ by

$$C_L / C_{L_{MD}} = \sqrt{C_{D_L}/(2-C_{D_L})} \quad (25)$$

In this case the optimum C_L for given M is found by substitution of Eq. (18) into Eq. (25), resulting in

$$C_L / C_{L_{MD}} = 1/\sqrt{1-2\eta_T} \quad (26)$$

The derivative C_{D_L} generally has a numerical value between 0 and 2, and must not be confused with the lift-dependent drag coefficient.

Propulsion Derivatives

In classical analytical solutions of flight mechanical problems, aircraft categories are usually subdivided into propeller aircraft, with η = const (i.e., $\eta_M = \eta_T = 0$), and "jet aircraft," with η proportional to the Mach number ($\eta_M = 1$, $\eta_T = 0$). From actual aircraft performances it is found that even for pure turbojets η_M is not equal to 1, but about 0.8, typically.

For turbofan engines operating at high-subsonic speeds typical values are between 0.5 and 0.8. As the value of η_M decreases with decreasing specific thrust, it appears that there is a large range of aircraft with high-bypass engines to which the classical criteria for optimum cruising are not applicable. These aircraft, however, are covered by the present analysis.

The derivative η_M can be related to the corrected TSFC, using Eq. (1) with $C_T = F/T$,

$$\eta_M = 1 - \frac{M}{C_T/\sqrt{\theta}}\left(\frac{\partial C_T/\sqrt{\theta}}{\partial M}\right)_{T/\delta} \quad (27)$$

A first approximation for η_M may be obtained by assuming a linear relationship between the corrected TSFC and the Mach number,

$$\frac{C_T}{\sqrt{\theta}} = C_1 + C_2 M \rightarrow \eta_M = \frac{C_1/C_2}{M + C_1/C_2} \quad (28)$$

Alternatively, an exponential equation can be used,

$$C_T/\sqrt{\theta} = C_3 M^n \rightarrow \eta_M = 1 - n \qquad (0<n<1) \quad (29)$$

It can be shown that in this equation the exponent n is related to the engine specific thrust.[7] Within the range of interest for cruise performance optimization the variation of η_M with M is usually small, as may be concluded from the nearly linear curves for constant T/δ in Fig. 1 and from Eq. (29) (provided n is constant).

The derivative η_T is normally close to zero for modern turbojet and turbofan engines operating at cruise conditions. In Fig. 1 this appears from the almost horizontal slope of the curves for constant M at high-subsonic speeds. Note also that η_T may be positive or negative. Within the range of normal cruise ratings the overall powerplant efficiency is therefore almost independent of the corrected thrust. Generally this

Table 1 Survey of general and special conditions for maximum *P*

Condition	General case (ref. Fig. 3)	$\eta_T = 0$ (turbojet and turbofan): General case	$\eta_T = 0$ (turbojet and turbofan): $\eta_M = 1$[a] (ref. Fig. 2)
Given M	$C_{D_L} = \frac{1}{1-\eta_T}$ (I)	$C_{D_L} = 1$	$C_{D_L} = 1$ (I)
Given C_L	$C_{D_M} = \frac{2\eta_T + \eta_M}{1-\eta_T}$ (II)	$C_{D_M} = \eta_M$[b]	$C_{D_M} = 1$[b] (II)
Constraint on W/δ	$2C_{D_L} - C_{D_M} = 2 - \frac{\eta_M}{1-\eta_T}$ (III)	$2C_{D_L} - C_{D_M} = 2 - \eta_M$	$2C_{D_L} - C_{D_M} = 1$ (III)
Constraint on T/δ	$C_{D_L}(2+\eta_M) - C_{D_M} = 2$ (IV)	$C_{D_L}(2+\eta_M) - C_{D_M} = 2$ (IV)	$3C_{D_L} - C_{D_M} = 2$ (IV)
Unconstrained optimum	I and II combined (point A)	$C_{D_L} = 1$; $C_{D_M} = \eta_M$[b]	$C_{D_L} = C_{D_M} = 1$[b] (point A′)

[a] $C_T/\sqrt{\theta}$ = const (i.e., independent of M). [b] Optimum drag rise.

Table 2 Conditions for maximum *P* for a drag polar without compressibility effects ($C_{D_M} = 0$)

Condition	$\eta_T = 0$ (turbojet/turbofan): $0 < \eta_M < 1$	$\eta_T = 0$ (turbojet/turbofan): $\eta_M = 1$[a]	$\eta_T \neq 0$ (turboprop): $\eta_M = 0$[b]
Given M	$C_{D_L} = 1$	$C_{D_L} = 1$	$C_{D_L} = \frac{1}{1-\eta_T}$
Constraint on W/δ	$C_{D_L} = 1 - \frac{1}{2}\eta_M$	$C_{D_L} = \frac{1}{2}$	$C_{D_L} = 1$
Constraint on T/δ	$C_{D_L} = \frac{2}{2+\eta_M}$	$C_{D_L} = \frac{2}{3}$	Not relevant

[a] $C_T/\sqrt{\theta}$ = const. [b] η = const (independent of M).

observation does not apply to turboprop engine/propeller combinations. It should also be noted that Eq. (23) is meaningless for propeller aircraft, since the constraint $T/\delta = \text{const}$ has no interpretation in the form of an operational constraint (e.g., on power setting).

Special Solutions

The general conditions for constrained and unconstrained maximum specific range, as derived in the previous sections, are summarized in Tables 1 and 2. These conditions hold for turbojet, turbofan, and turboprop aircraft. Recalling the observations on the logarithmic derivatives, some special cases can be considered, which will be discussed briefly.

Case $\eta_T = 0$

Since η_T is close to zero for most turbofan and turbojet engines, the general character of the analysis in this paragraph is impaired only by the fact that propeller aircraft are excluded. The following simplified solutions are obtained:

1) For given M, from Eq. (18): $C_{D_L} = 1$, or $\partial C_D/\partial C_L = C_D/C_L$. This corresponds to flying at the maximum L/D ratio for the Mach number under consideration.

2) For given C_L, from Eq. (19): $C_{D_M} = \eta_M$, or $\partial C_D/\partial M = \eta_M C_D/M$. This result proves that the maximum range parameter is found for a Mach number in the drag rise, provided $\eta_M > 0$.

A typical order of magnitude is $\partial C_D/\partial M = 0.020\text{-}0.025$, resulting in a Mach number some points below the drag-critical Mach number (according to the definition $\partial C_D/\partial M = 0.10$).

The *unconstrained maximum* is obtained by combination of the two conditions above as illustrated in the graphical presentation of Fig. 4. The same solution may also be obtained with Fig. 2 by intersecting the curve $C_{D_L} = 1$ (line I) with a line $C_{D_M} = \eta_M$ to be derived from the slope of $\log C_D$ vs $\log M$.

Case $\eta_M = 1$ (with $\eta_T = 0$)

This class of solutions corresponds to the classical case of "pure turbojet" aircraft. All conditions are now expressed in terms of the aerodynamic logarithmic derivatives C_{D_L} and C_{D_M} only, as indicated by the lines I-IV in Fig. 2.

The unconstrained optimum is the intersection point A′, yielding the absolute maximum of $M(L/D)$. In the same figure the condition for maximum L/D is also indicated (line V), for which one can derive from $L/D = f(C_L, M)$, at constant W/δ (cf. Ref. 4, Chap. 2),

$$2C_{D_L} - C_{D_M} = 2 \tag{30}$$

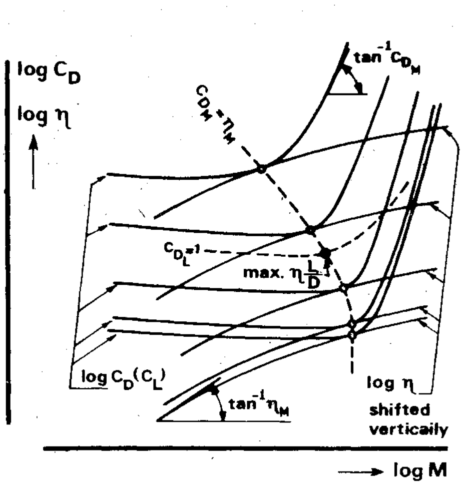

Fig. 4 Graphical solution of approximate optimum unconstrained cruise condition ($\eta_T = 0$).

For low-subsonic Mach numbers ($C_{D_M} \approx 0$) this curve merges into the line $C_{D_L} = 1$ (curve I).

Case $C_{D_M} = 0$ (with $\eta_T = 0$)

Another set of solutions is found if the compressibility effects can be neglected. From inspection of Eq. (19) the necessary condition $\eta_M = 0$ is found, but this cannot be satisfied for turbojet and turbofan engines. Therefore no unconstrained optimum solution can be obtained, and the range parameter increases monotonically with speed and altitude.

For the case of active constraints on W/δ or T/δ, however, solutions can be obtained from Eqs. (21) and (23), respectively, as illustrated in Table 2 for different values of η_M. For the case of a parabolic drag polar (the same for all Mach numbers) it appears possible to find explicit results in terms of the optimum values for C_L and M. After substitution of Eq. (25) into Eqs. (21) and (23), respectively, the following results are found:

1) Constraint on W/δ (or $C_L M^2$)

$$\frac{C_L}{C_{L_{MD}}} = \sqrt{\frac{2-\eta_M}{2+\eta_M}}; \quad \frac{M}{M_{MD}} = \sqrt{\frac{C_{L_{MD}}}{C_L}} = \sqrt[4]{\frac{2+\eta_M}{2-\eta_M}} \tag{31}$$

where M_{MD} is obtained from

$$M_{MD} = \sqrt{\frac{2W/\delta}{\gamma p_0 M^2 S C_{L_{MD}}}} \tag{32}$$

2) Constraint on T/δ (or $C_D M^2$)

$$\frac{C_L}{C_{L_{MD}}} = \frac{1}{\sqrt{1+\eta_M}}; \quad \frac{M}{M_{MD}} = \sqrt{\frac{C_{D_{MD}}}{C_D}} = \sqrt{\frac{2(1+\eta_M)}{2+\eta_M}} \tag{33}$$

where the following relationship has been used

$$\frac{C_D}{C_{D_{MD}}} = \tfrac{1}{2}\left[1 + \left(\frac{C_L}{C_{L_{MD}}}\right)^2\right] \tag{34}$$

and M_{MD} is obtained from

$$M_{MD} = \sqrt{\frac{2T/\delta}{\gamma p_0 M^2 S C_{D_{MD}}}} \tag{35}$$

The well-known classical results for the ratios of $C_L/C_{L_{MD}}$ and M/M_{MD} (cf. Ref. 2) can be obtained directly from Eqs. (31) and (33) by substituting for propeller aircraft:

$$\eta_M = 0 \rightarrow C_L/C_{L_{MD}} = 1$$

and for "pure turbojet" aircraft:

$$\eta_M = 1 \rightarrow C_L/C_{L_{MD}} = 1/\sqrt{3} \text{ for } W/\delta = \text{const}$$

and

$$C_L/C_{L_{MD}} = 1/\sqrt{2} \text{ for } T/\delta = \text{const}$$

For turbojet aircraft these results are usually interpreted as an optimum flight Mach number equal to $\sqrt[4]{3} M_{MD}$ for given altitude and $\sqrt[4]{2} M_{MD}$ for given engine rating, respectively. However, Eq. (35) shows that in the case of given T/δ it is more logical to relate the cruise condition to M_{MD} as determined by the horizontal equilibrium (thrust = drag), rather than relating it to M_{MD} at an unknown altitude to be determined. In that case Eq. (33) yields $M/M_{MD} = 2/\sqrt{3}$.

The present analysis for turbofan engines indicates that optimum cruise speeds (for $\eta_M < 1$) are closer to the minimum drag speed, as compared with the results from classical flight mechanics. This is in accordance with actual operational practice, where aircraft are flown at 10-15% above M_{MD}.

Case $\eta_M = 0$

This case applies to propeller aircraft if the total engine efficiency and propeller efficiency are considered independent of M. If in addition $\eta_T = 0$, the overall engine efficiency is independent of the power setting and the altitude, resulting in $C_{D_L} = 1$ for optimum cruising, i.e., flight at the minimum drag speed for each altitude. For turboprop engines the overall efficiency increases with altitude and hence the specific range. The optimum flight condition is therefore determined by a constraint on the power rating at altitude. This case has been treated adequately in classical flight mechanics.

Conclusion

The conditions for maximum specific range have been considered for a given aircraft/engine combination. The optimum flight conditions have been derived, taking into account the effects of compressibility on the airplane drag and the effects of engine rating, altitude, and speed on the engine characteristics. Special cases are obtained by introducing approximations for the drag and/or engine characteristics. The optimum cruise condition is either a fixed point in the generalized drag polars (for the unconstrained case) or a combination of a fixed curve in this polar plot with constraints on T/δ or W/δ. For turbojet and turbofan aircraft the unconstrained optimum cruise condition is essentially located in the drag rise. Therefore, the classical criteria are not accurate enough for this class of high-subsonic aircraft.

The present method gives a useful tool for performance optimization in the preliminary design stage, particularly if a reliable estimate of the compressibility drag is available. The Appendix illustrates how the method may be applied in practice. It is suggested that the present results can also be useful for on-board computation of optimum flight conditions for long-range cruising. It can be shown[5,6] that conditions for minimum direct operating costs or wind effects can also be incorporated into the present theory.

Appendix: Numerical Example

An illustration of the practical use of the present analysis will be given for a hypothetical high-subsonic medium-haul trijet for 185 passengers (all-economy), characterized by a max takeoff weight of 1100 kN (247,200 lb) and $S = 200$ m^2 (2150 ft^2). The three engines are high-bypass-ratio turbofans in the ~100 kN (22,000 lb) sea level takeoff thrust class. Figure 2 is used for the drag polars, approximated for $M = 0.8$, for example, by $C_D = 0.015 + 0.049C_L^2$ (for $0.2 < C_L < 0.55$).

The overall engine efficiency is shown in Fig. 1; the derivation of one point will be demonstrated. For $M = 0.794$ ($\log M = -0.1$) at 10,675 m (35,000 ft) altitude, $\delta = 0.235$; the engine brochure states $C_T = 18.26$ mg/Ns (0.645 lb/lb·h). From the definition of η [Eq. (1)] we have for $V = 235$ m/s (770 ft/s) and $H = 43.14$ J/mg (18,550 Btu/lb): $\eta = 0.298$ or $\log \eta = -0.525$. For a 90% maximum cruise thrust of 20 kN (4500 lb) per engine we find $T/\delta T_{to} = 0.87$, hence $\log(T/\delta T_{to}) = -0.061$. This point has been marked in Fig. 1. From the slopes of the curves for constant M and constant T/δ it is found that $\eta_M = 0.52$ and $\eta_T = 0.05$.

The unconstrained optimum cruise point according to Eqs. (18) and (19) is $C_{D_L} = 1.05$ and $C_{D_M} = 0.65$. From the slopes of the curves in Fig. 2 it is found that $\log M = -0.086$ ($M = 0.82$) and $\log C_L = -0.31$ ($C_L = 0.49$). Hence $C_D = 0.0267$ and $L/D = 18.27$. The optimum altitude is obtained from Eq. (3): $\delta = 0.221$, i.e., 11,000 m (36,000 ft) altitude.

For an AUW of 1050 kN (236,000 lb) the thrust required is $1050 \div 18.27 = 57.47$ kN, or 19.16 kN (4308 lb) per engine, hence $\log(T/\delta T_{to}) = -0.0525$. In Fig. 2 we read $\eta = 0.305$, and the maximum range parameter is $P = \eta L/D = 5.57$.

The effect of an altitude constraint is illustrated by assuming that the cruise altitude is constrained to 9150 m (30,000 ft), or $\delta = 0.297$, and $W/\delta = 3533$ kN (794,000 lb). Equation (3) yields $C_L M^2 = 0.245$, or

$$\log C_L + 2\log M = -0.61 \tag{A1}$$

Using the values for η_M and η_T given above, Eq. (21) becomes

$$2C_{D_L} - C_{D_M} = 1.45 \tag{A2}$$

A plot of Eqs. (A1) and (A2) in Fig. 2 provides an intersection at $\log M = -0.101$ ($M = 0.792$), $C_L = 0.39$, $C_D = 0.022$, and hence $L/D = 17.57$. The thrust required per engine is 19.92 kN (4477 lb), and we obtain $\eta = 0.295$ from Fig. 1. The range parameter $P = 5.18$ indicates a penalty of 7% as compared to the maximum value.

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